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Zeros of some bi-orthogonal polynomials

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Abstract
Ercolani and McLaughlin have recently shown that the zeros of the bi-orthogonal polynomials with the weight $w(x, y) = \exp[-(V_1(x) + V_2(y) + 2cxy)/2]$, relevant to a model of two coupled Hermitian matrices, are real and simple. We show that their argument applies to the more general case of the weight $(w_1 * w_2 * \cdots * w_j)(x, y)$, a convolution of several weights of the same form. This general case is relevant to a model of several Hermitian matrices coupled in a chain. Their argument also works for more general weights such as $W(x, y) = \mathrm{e}^{-x-y}/(x + y)$, $0 \leqslant x, y < \infty$, and for a convolution of several such weights.

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1. Introduction

For a weight function $w(x, y)$ such that all the moments

$$M_{i,j} := \int w(x, y) x^i y^j \, \mathrm{d}x \, \mathrm{d}y \tag{1.1}$$

exist and

$$D_n := \det[M_{i,j}]_{i,j=0,1,\ldots,n} \neq 0 \tag{1.2}$$

for all $n \geqslant 0$, unique monic polynomials $p_n(x)$ and $q_n(x)$ of degree n exist satisfying the bi-orthogonality relations (a polynomial is called monic when the coefficient of the highest degree is one)

$$\int w(x, y) p_n(x) q_m(y) \, \mathrm{d}x \, \mathrm{d}y = h_n \delta_{mn}. \tag{1.3}$$

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Just like the orthogonal polynomials they can be expressed as determinants, e.g.

$$p_n(x) = \frac{1}{D_{n-1}} \det \begin{bmatrix} M_{0,0} & \dots & M_{0,n-1} & 1 \\ M_{1,0} & \dots & M_{1,n-1} & x \\ \vdots & \vdots & \vdots & \vdots \\ M_{n,0} & \dots & M_{n,n-1} & x^n \end{bmatrix} \tag{1.4}$$

and have integral representations, e.g.

$$p_n(x) \propto \int \Delta_n(\boldsymbol{x})\Delta_n(\boldsymbol{y}) \prod_{j=1}^{n} (x - x_j) w(x_j, y_j)\, \mathrm{d}x_j\, \mathrm{d}y_j \tag{1.5}$$

$$\Delta_n(\boldsymbol{x}) := \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i) \qquad \Delta_n(\boldsymbol{y}) := \prod_{1 \leqslant i < j \leqslant n} (y_j - y_i). \tag{1.6}$$

From limited numerical evidence for the weights

(i) $w(x, y) = \sin(\pi x y), \quad 0 \leqslant x, y \leqslant 1;$
(ii) $w(x, y) = |x - y|, \quad -1 \leqslant x, y \leqslant 1;$
(iii) $w(x, y) = \mathrm{e}^{-x-y}/(x + y), \quad 0 \leqslant x, y < \infty;$
(iv) $w(x, y) = \exp(-x^2 - y^2 - cxy), \quad -\infty < x, y < \infty, \quad 0 < c < 2;$

one might think that the zeros of the bi-orthogonal polynomials are real, simple, lie respectively in the x- or y-support of $w(x, y)$, interlace for successive $n, \dots$.

Alas, this is not true in general as seen by the following example due to Pierre Deligne. If one takes

$$w(x, y) = u(x, y) + v(x, y) \tag{1.7}$$

$$u(x, y) = \begin{cases} \delta(x - y) & -1 \leqslant x, y \leqslant 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

$$v(x, y) = \tfrac{1}{8}[\delta(x - 1)\delta(y + 2) + \delta(x + 1)\delta(y - 2)]. \tag{1.9}$$

Then $p_3(x)$ and $q_3(x)$ have complex zeros.

However, Ercolani and Mclaughlin have recently [1] shown that with the weight function

$$w(x, y) = \exp\left[-\tfrac{1}{2}V_1(x) - \tfrac{1}{2}V_2(y) - cxy\right] \tag{1.10}$$

$(-\infty < x, y < \infty)$, V_1 and V_2 polynomials of positive even degree, c a small non-zero real constant, all the zeros of the bi-orthogonal polynomials $p_n(x)$ and $q_n(x)$ are real and simple.

In this brief paper we will show that their argument works for the following general case encountered in random Hermitian matrices coupled in a linear chain. Let $V_j(x)$, $1 \leqslant j \leqslant p$, be polynomials of positive even degree and c_j, $1 \leqslant j < p$, be small real constants, none of them being zero ('small' so that all the moments $M_{i,j}$ defined below, equation (1.13), exist). Further let

$$w_k(x, y) := \exp\left[-\tfrac{1}{2}V_k(x) - \tfrac{1}{2}V_{k+1}(y) - c_k xy\right] \tag{1.11}$$

$$\left(w_{i_1} * w_{i_2} * \cdots * w_{i_k}\right)(\xi_1, \xi_{k+1}) := \int w_{i_1}(\xi_1, \xi_2) w_{i_2}(\xi_2, \xi_3) \dots w_{i_k}(\xi_k, \xi_{k+1})\, \mathrm{d}\xi_2 \dots \mathrm{d}\xi_k. \tag{1.12}$$

Moreover, assume that for all $i, j \geqslant 0$

$$M_{i,j} := \int x^i (w_1 * w_2 * \cdots * w_{p-1})(x,\ y) y^j\, \mathrm{d}x\, \mathrm{d}y \tag{1.13}$$

exist.

Theorem. *Then monic polynomials $p_j(x)$ and $q_j(x)$ can be uniquely defined by*

$$\int p_j(x)(w_1 * w_2 * \cdots * w_{p-1})(x, y)q_k(y)\,\mathrm{d}x\,\mathrm{d}y = h_j\delta_{jk} \tag{1.14}$$

and all the zeros of $p_j(x)$ and of $q_j(x)$ are real and simple.

The same argument works for any weight $W(x, y)$ such that $\det[W(x_i, y_j)]_{i,j=1,\ldots,n} > 0$ for $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$ and moments $M_{i,j} = \int W(x, y)x^i y^j\,\mathrm{d}x\,\mathrm{d}y$ exist for all $i, j \geqslant 0$. For example, if $W(x, y) = \mathrm{e}^{-x-y}/(x + y)$, $0 \leqslant x, y < \infty$, then monic polynomials $p_j(x)$ can be uniquely defined by

$$\int_0^\infty p_j(x)W(x, y)p_k(y)\,\mathrm{d}x\,\mathrm{d}y = h_j\delta_{jk} \tag{1.15}$$

(here $W(x, y)$ is symmetric in x and y so that $p_j(x) = q_j(x)$) and all the zeros of $p_j(x)$ are real, simple and non-negative.

The weight function $W(x, y) = (w_1 * w_2 * \cdots * w_p)(x, y)$ with equations (1.11) and (1.12) is relevant to a model of p Hermitian matrices coupled in a chain [2]. In fact all the correlation functions of the eigenvalues of these matrices can be expressed as determinants whose elements are combinations of $P_{i,j}(x)$ and $Q_{i,j}(x)$, equations (2.1)–(2.6), where $p_j(x)$ and $q_j(x)$ are polynomials bi-orthogonal with respect to this $W(x, y)$. For example, the eigenvalue density of the matrices at the ends of the chain is related to the density of the zeros of these bi-orthogonal polynomials. A knowledge of the distribution of the zeros of $P_{i,j}(x)$ and $Q_{i,j}(x)$ can thus be very useful to study the correlation functions in the above model.

2. Results and proofs

Here we essentially follow section 3 of [1]. With any monic polynomials $p_j(x)$ and $q_j(x)$ of degree j, let us write

$$P_{1,j}(x) := p_j(x) \tag{2.1}$$

$$P_{i,j}(x) := \int p_j(\xi)U_{Li}(\xi, x)\,\mathrm{d}\xi \qquad 1 < i \leqslant p \tag{2.2}$$

$$U_{Li}(\xi, x) := (w_1 * w_2 * \cdots * w_{i-1})(\xi, x) \qquad 1 < i \leqslant p \tag{2.3}$$

$$Q_{p,j}(x) := q_j(x) \tag{2.4}$$

$$Q_{i,j}(x) := \int U_{Ri}(x, \xi)q_j(\xi)\,\mathrm{d}\xi \qquad 1 \leqslant i < p \tag{2.5}$$

$$U_{Ri}(x, \xi) := (w_i * w_{i+1} * \cdots * w_{p-1})(x, \xi) \qquad 1 \leqslant i < p \tag{2.6}$$

Note that $P_{i,j}(x)$ and $Q_{i,j}(x)$ are not necessarily polynomials.

Lemma 1. *For $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$*

$$\det[w_i(x_j, y_k)]_{j,k=1,\ldots,n} > 0. \tag{2.7}$$

This is essentially equation (40) of [1]. This can also be seen as follows. Let X and Y be two $n \times n$ diagonal matrices with diagonal elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively. Then the integral of $\exp[-c\,\mathrm{tr}\,(UXU^{-1}Y)]$ over the $n \times n$ unitary matrices U is given by [3]

$$K\frac{\det[\exp(-c\,x_i y_j)]_{i,j=1,\ldots,n}}{\Delta_n(\boldsymbol{x})\Delta_n(\boldsymbol{y})} \tag{2.8}$$

where K is a positive constant depending on c and n. Hence

$$\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}[V_i(x_j)+V_{i+1}(y_j)]\right\}\int \mathrm{d}U\,\mathrm{e}^{-c_i \operatorname{tr} UXU^{-1}Y} = K\frac{\det[w_i(x_j,y_k)]_{j,k=1,\ldots,n}}{\Delta_n(\boldsymbol{x})\Delta_n(\boldsymbol{y})}. \tag{2.9}$$

The left-hand side is evidently positive while on the right hand side the denominator is positive since $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$. From this equation (2.7) follows.

Lemma 2. *For* $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$,

$$\det\left[\left(w_{i_1} * w_{i_2} * \cdots * w_{i_\ell}\right)(x_j, y_k)\right]_{j,k=1,\ldots,n} > 0. \tag{2.10}$$

Proof. Binet–Cauchy formula tells us that [4]

$$\det\left[\left(w_{i_1} * w_{i_2}\right)(x_j, y_k)\right]_{j,k=1,\ldots,n}$$

is equal to

$$\int_{\xi_1\leqslant\xi_2\leqslant\cdots\leqslant\xi_n} \det\left[w_{i_1}(x_j,\xi_k)\right]_{j,k=1,\ldots,n}\cdot\det\left[w_{i_2}(\xi_j,y_k)\right]_{j,k=1,\ldots,n}\mathrm{d}\xi_1\ldots\mathrm{d}\xi_n. \tag{2.11}$$

By lemma 1 the integrand is everywhere positive, so lemma 2 is proved for the case $\ell = 2$. The proof is now completed by induction on ℓ, again using the Binet–Cauchy formula. □

Lemma 3. *For any monic polynomial* $p_j(x)$ *of degree* j, $P_{i,j}(x)$, $1 \leqslant i \leqslant p$, *may have at most* j *distinct real zeros. Similarly, for any monic polynomial* $q_j(x)$ *of degree* j, $Q_{i,j}(x)$, $1 \leqslant i \leqslant p$, *may have at most* j *distinct real zeros.*

Proof. Let, if possible, $z_1 < z_2 < \cdots < z_m$, $m > j$, be the distinct real zeros of $P_{i,j}(x)$. Since

$$P_{i,j}(x) = \sum_{k=0}^{j} a_k T_{i,k}(x) \tag{2.12}$$

with

$$T_{i,k}(x) := \int \xi^k U_{Li}(\xi, x)\,\mathrm{d}\xi \tag{2.13}$$

$$P_{i,j}(z_\ell) = 0 \qquad \ell = 1, 2, \ldots, m \quad m > j \tag{2.14}$$

implies that

$$0 = \det\begin{bmatrix} T_{i,0}(z_1) & T_{i,1}(z_1) & \ldots & T_{i,j}(z_1) \\ \ldots & \ldots & \ldots & \ldots \\ T_{i,0}(z_{j+1}) & T_{i,1}(z_{j+1}) & \ldots & T_{i,j}(z_{j+1}) \end{bmatrix}$$

$$= \int \det\begin{bmatrix} U_{Li}(\xi_1,z_1) & \xi_2 U_{Li}(\xi_2,z_1) & \ldots & \xi_{j+1}^{j} U_{Li}(\xi_{j+1},z_1) \\ \ldots & \ldots & \ldots & \ldots \\ U_{Li}(\xi_1,z_{j+1}) & \xi_2 U_{Li}(\xi_2,z_{j+1}) & \ldots & \xi_{j+1}^{j} U_{Li}(\xi_{j+1},z_{j+1}) \end{bmatrix}\mathrm{d}\xi_1\ldots\mathrm{d}\xi_{j+1}$$

$$= \int \xi_2\xi_3^2\ldots\xi_{j+1}^{j}\det\left[U_{Li}(\xi_k,z_\ell)\right]_{k,\ell=1,\ldots,j+1}\mathrm{d}\xi_1\ldots\mathrm{d}\xi_{j+1} \tag{2.15}$$

or

$$\int \det\left[U_{Li}(\xi_k,z_\ell)\right]_{k,\ell=1,\ldots,j+1}\cdot\det\left[\xi_k^{\ell-1}\right]_{k,\ell=1,\ldots,j+1}\mathrm{d}\xi_1\ldots\mathrm{d}\xi_{j+1} = 0 \tag{2.16}$$

in contradiction to lemma 2. Thus m cannot be greater than j. □

The proof for $Q_{i,j}(x)$ is similar.

Lemma 4. *Let the real constants $c_1, \ldots, c_{p-1}$, none of them being zero, be such that $M_{i,j}$ defined by equation (1.13) exist for all i, $j \geqslant 0$. Then*

$$D_n := \det[M_{i,j}]_{i,j=1,\ldots,n} \neq 0 \tag{2.17}$$

for any $n \geqslant 0$.

Proof. Let, if possible, $D_n = 0$ for some n. Then $\sum_{j=0}^{n} M_{i,j} q_j = 0$, q_j not all zero, and

$$\int x^i U_{Lp}(x, y) \sum_{j=0}^{n} q_j y^j \,\mathrm{d}x\,\mathrm{d}y = 0 \qquad i = 0,\ 1, \ldots, n \tag{2.18}$$

or

$$\int p_i(x) U_{Lp}(x, y) \sum_{j=0}^{n} q_j y^j \,\mathrm{d}x\,\mathrm{d}y = 0 \tag{2.19}$$

for any polynomial $p_i(x)$ of degree $i \leqslant n$. But

$$\int U_{Lp}(x, y) \sum_{j=0}^{n} q_j y^j \,\mathrm{d}y \tag{2.20}$$

has at most n distinct real zeros (lemma 3). So one can choose $p_i(x)$ such that

$$p_i(x) \int U_{Lp}(x, y) \sum_{j=0}^{n} q_j y^j \,\mathrm{d}y > 0 \tag{2.21}$$

in contradiction to equation (2.19). So $D_n \neq 0$ and bi-orthogonal polynomials $p_j(x)$, $q_j(x)$ exist (see equations (1.4) and (1.5)). $\square$

Lemma 5. *Let $p_j(x), q_j(x)$ be the bi-orthogonal polynomials, equation (1.14); or with the definitions (2.1)–(2.6)*

$$\int P_{i,j}(x) Q_{i,k}(x) \,\mathrm{d}x = h_j \delta_{jk} \qquad 1 \leqslant i \leqslant p. \tag{2.22}$$

Then $P_{i,j}(x)$ has at least j real distinct zeros of odd multiplicity. So does have $Q_{i,j}(x)$.

Proof. Assuming $m > 0$, let, if possible, $z_1 < z_2 < \cdots < z_m$, $m < j$, be the only real zeros of $P_{i,j}(x)$ of odd multiplicity. Set

$$R(x) = \det \begin{bmatrix} Q_{i,0}(x) & Q_{i,1}(x) & \ldots & Q_{i,m}(x) \\ Q_{i,0}(z_1) & Q_{i,1}(z_1) & \ldots & Q_{i,m}(z_1) \\ \vdots & \vdots & & \vdots \\ Q_{i,0}(z_m) & Q_{i,1}(z_m) & \ldots & Q_{i,m}(z_m) \end{bmatrix} \tag{2.23}$$

$$= \int U_{Ri}(x, \xi) \sum_{k=0}^{m} \alpha_k \xi^k \,\mathrm{d}\xi \tag{2.24}$$

with some constants α_k depending on $z_1, \ldots, z_m$.

Since $m < j$, the bi-orthogonality gives

$$\int P_{i,j}(x) R(x) \,\mathrm{d}x = 0. \tag{2.25}$$

However, $R(x)$ can also be written as

$$R(x) = \int \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1)\xi_1 & \dots & U_{Ri}(x,\xi_m)\xi_m^m \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1)\xi_1 & \dots & U_{Ri}(z_1,\xi_m)\xi_m^m \\ \vdots & \vdots & & \vdots \\ U_{Ri}(z_m,\xi_0) & U_{Ri}(z_m,\xi_1)\xi_1 & \dots & U_{Ri}(z_m,\xi_m)\xi_m^m \end{bmatrix} \mathrm{d}\xi_0\,\mathrm{d}\xi_1 \dots \mathrm{d}\xi_m$$

$$= \int \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1) & \dots & U_{Ri}(x,\xi_m) \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1) & \dots & U_{Ri}(z_1,\xi_m) \\ \vdots & \vdots & & \vdots \\ U_{Ri}(z_m,\xi_0) & U_{Ri}(z_m,\xi_1) & \dots & U_{Ri}(z_m,\xi_m) \end{bmatrix} \xi_1\xi_2^2\dots\xi_m^m\,\mathrm{d}\xi_0\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_m$$

$$= \frac{1}{(m+1)!}\int \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1) & \dots & U_{Ri}(x,\xi_m) \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1) & \dots & U_{Ri}(z_1,\xi_m) \\ \vdots & \vdots & & \vdots \\ U_{Ri}(z_m,\xi_0) & U_{Ri}(z_m,\xi_1) & \dots & U_{Ri}(z_m,\xi_m) \end{bmatrix}$$

$$\times \prod_{0\leqslant r<s\leqslant m} (\xi_s-\xi_r)\,\mathrm{d}\xi_0\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_m$$

$$= \int_{\xi_0\leqslant\xi_1\leqslant\cdots\leqslant\xi_m} \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1) & \dots & U_{Ri}(x,\xi_m) \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1) & \dots & U_{Ri}(z_1,\xi_m) \\ \vdots & \vdots & & \vdots \\ U_{Ri}(z_m,\xi_0) & U_{Ri}(z_m,\xi_1) & \dots & U_{Ri}(z_m,\xi_m) \end{bmatrix}$$

$$\times \prod_{0\leqslant r<s\leqslant m} (\xi_s-\xi_r)\,\mathrm{d}\xi_0\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_m. \tag{2.26}$$

Thus $R(x)$ is represented as an integral whose integrand has a fixed sign determined by the relative ordering of the numbers $x, z_1, z_2, \dots, z_m$ (lemma 2). It thus follows that $R(x)$ changes sign when x passes through any of the points z_k, $k = 1, \dots, m$, and at no other value of x. In other words, $z_1, \dots, z_m$ are the only real zeros of $R(x)$ having an odd multiplicity.
And therefore $P_{i,j}(x)R(x)$ has a constant sign, so that

$$\int P_{i,j}(x)R(x)\,\mathrm{d}x \neq 0 \tag{2.27}$$

in contradiction to (2.25).

If $P_{i,j}(x)$ has no real zeros of odd multiplicity ($m = 0$), then $R(x) = \int U_{Ri}(x,\xi)\,\mathrm{d}\xi$ is everywhere positive which contradicts equation (2.25). □

The proof for $Q_{i,j}(x)$ is similar.

As a consequence, we have the integral representations of $P_{i,j}(x)$ for $i > 1$ and of $Q_{i,j}(x)$ for $i < p$ involving their respective zeros

$$P_{i,j}(x) \propto \int \det \begin{bmatrix} U_{Li}(\xi_0,x) & U_{Li}(\xi_1,x) & \dots & U_{Li}(\xi_j,x) \\ U_{Li}(\xi_0,z_1) & U_{Li}(\xi_1,z_1) & \dots & U_{Li}(\xi_j,z_1) \\ \vdots & \vdots & & \vdots \\ U_{Li}(\xi_0,z_j) & U_{Li}(\xi_1,z_j) & \dots & U_{Li}(\xi_j,z_j) \end{bmatrix}$$

$$\times \prod_{0\leqslant r<s\leqslant j} (\xi_s-\xi_r)\,\mathrm{d}\xi_0\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_j \tag{2.28}$$

$$Q_{i,j}(x) \propto \int \det \begin{bmatrix} U_{Ri}(x,\xi_0) & U_{Ri}(x,\xi_1) & \ldots & U_{Ri}(x,\xi_j) \\ U_{Ri}(z_1,\xi_0) & U_{Ri}(z_1,\xi_1) & \ldots & U_{Ri}(z_1,\xi_j) \\ \vdots & \vdots & & \vdots \\ U_{Ri}(z_j,\xi_0) & U_{Ri}(z_j,\xi_1) & \ldots & U_{Ri}(z_j,\xi_j) \end{bmatrix}$$
$$\times \prod_{0\leqslant r<s\leqslant j} (\xi_s - \xi_r)\, \mathrm{d}\xi_0\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_j. \tag{2.29}$$

Lemmas 3 and 5 tell us that if $p_j(x)$ and $q_j(x)$ are bi-orthogonal polynomials satisfying equation (1.14), then $P_{i,j}(x)$ and $Q_{i,j}(x)$ each have exactly j distinct real zeros of odd multiplicity. In particular, the zeros of the bi-orthogonal polynomials $p_j(x) \equiv P_{1,j}(x)$ and $q_j(x) \equiv Q_{p,j}(x)$ are real and simple.

With a little more effort one can perhaps show that all the real zeros of $P_{i,j}(x)$ and of $Q_{i,j}(x)$ are simple. Other zeros, if any, must be complex. Whether the zeros of $p_j(x)$ $(q_j(x))$ interlace for successive j, remains an open question.

3. Bi-orthogonal polynomials with another weight

For the weight $W(x,y) = \mathrm{e}^{-x-y}/(x+y)$, $0 \leqslant x, y < \infty$, one can say as follows:

Lemma 1′. *One has [4]:*

$$\det[W(x_j,y_k)]_{j,k=1,\ldots,n} = \exp\left[-\sum_{j=1}^{n}(x_j+y_j)\right] \Delta_n(\boldsymbol{x})\Delta_n(\boldsymbol{y}) \prod_{j,k=1}^{n}(x_j+y_k)^{-1} \tag{3.1}$$

which is evidently positive for $0 \leqslant x_1 < x_2 < \cdots < x_n$, $0 \leqslant y_1 < y_2 < \cdots < y_n$.

Lemma 3′. *For any monic polynomial* $p_j(x)$ *of degree* j, $P_j(x) := \int_0^\infty W(x,y)p_j(y)\,\mathrm{d}y$ *has at most* j *distinct real non-negative zeros.*

In the proof of lemma 3, replace equations (2.12)–(2.16) by

$$P_j(x) = \sum_{k=0}^{j} a_k T_k(x) \tag{3.2}$$

$$T_k(x) = \int_0^\infty \xi^k W(\xi,x)\,\mathrm{d}\xi \tag{3.3}$$

$$P_j(z_\ell) = 0 \qquad \ell = 1,2,\ldots,m, \quad m > j \tag{3.4}$$

$$0 = \det \begin{bmatrix} T_0(z_1) & T_1(z_1) & \ldots & T_j(z_1) \\ \ldots & \ldots & \ldots & \ldots \\ T_0(z_{j+1}) & T_1(z_{j+1}) & \ldots & T_j(z_{j+1}) \end{bmatrix}$$
$$= \int_0^\infty \det \begin{bmatrix} W(\xi_1,z_1) & \xi_2 W(\xi_2,z_1) & \ldots & \xi_{j+1}^j W(\xi_{j+1},z_1) \\ \ldots & \ldots & \ldots & \ldots \\ W(\xi_1,z_{j+1}) & \xi_2 W(\xi_2,z_{j+1}) & \ldots & \xi_{j+1}^j W(\xi_{j+1},z_{j+1}) \end{bmatrix} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{j+1}$$
$$= \int \xi_2\xi_3^2 \ldots \xi_{j+1}^j \det[W(\xi_k,z_\ell)]_{k,\ell=1,\ldots,j+1}\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{j+1} \tag{3.5}$$

or

$$\int \det[W(\xi_k,z_\ell)]_{k,\ell=1,\ldots,j+1} \cdot \det\left[\xi_k^{\ell-1}\right]_{k,\ell=1,\ldots,j+1} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{j+1} = 0 \tag{3.6}$$

in contradiction to lemma 1′. Thus m cannot be greater than j.

Lemma 4′. *With*

$$M_{i,j} := \int_0^\infty x^i W(x, y) y^j \, \mathrm{d}x \, \mathrm{d}y \tag{3.7}$$

$$D_n := \det[M_{i,j}]_{i,j=0,1,\ldots,n} \neq 0 \tag{3.8}$$

for any $n \geqslant 0$.

In the proof of lemma 4 replace everywhere $\int U_{Lp}(x, y) \ldots$ by $\int_0^\infty W(x, y) \ldots$.

Lemma 5′. *Let* $p_j(x)$ *be the (bi-orthogonal) polynomials satisfying*

$$\int_0^\infty W(x, y) p_j(x) p_k(y) \, \mathrm{d}x \, \mathrm{d}y = h_j \delta_{jk}. \tag{3.9}$$

Then $P_j(x) := \int_0^\infty W(x, y) p_j(y) \, \mathrm{d}y$ *and* $p_j(x)$ *each have at least* j *distinct real non-negative zeros of odd multiplicity.*

Let, if possible, $0 \leqslant z_1 < z_2 < \cdots < z_m$ $m < j$, be the only real non-negative zeros of $P_j(x)$ of odd multiplicity. Set $R(x) = \prod_{j=1}^m (x - z_j)$. Then as $m < j$, one has

$$\int_0^\infty P_j(x) R(x) \, \mathrm{d}x = 0. \tag{3.10}$$

But $P_j(x)$ and $R(x)$ change sign simultaneously as x passes through the values $z_1, \ldots, z_m$ and at no other real positive value. So the product $P_j(x)R(x)$ never changes sign, in contradiction to (3.10). Therefore $P_j(x)$ has at least j distinct real non-negative zeros of odd multiplicity.

To prove that $p_j(x)$ has at least j distinct real non-negative zeros let, if possible, $0 \leqslant z_1 < z_2 < \cdots < z_m$, $m < j$, be the only such zeros. Set

$$\begin{aligned} R(x) &= \det \begin{bmatrix} P_0(x) & P_1(x) & \ldots & P_m(x) \\ P_0(z_1) & P_1(z_1) & \ldots & P_m(z_1) \\ \vdots & \vdots & & \vdots \\ P_0(z_m) & P_1(z_m) & \ldots & P_m(z_m) \end{bmatrix} \\ &= \int_0^\infty W(x, \xi) \sum_{k=0}^m \alpha_k \xi^k \, \mathrm{d}\xi \end{aligned} \tag{3.11}$$

with some constants α_k depending on $z_1, \ldots, z_m$.

Since $m < j$, the bi-orthogonality gives

$$\int_0^\infty p_j(x) R(x) \, \mathrm{d}x = 0. \tag{3.12}$$

But

$$\begin{aligned} R(x) \propto \int_0^\infty \det & \begin{bmatrix} W(x, \xi_0) & W(x, \xi_1) & \ldots & W(x, \xi_m) \\ W(z_1, \xi_0) & W(z_1, \xi_1) & \ldots & W(z_1, \xi_m) \\ \vdots & \vdots & & \\ W(z_m, \xi_0) & W(z_m, \xi_1) & \ldots & W(z_m, \xi_m) \end{bmatrix} \\ & \times \prod_{0 \leqslant r < s \leqslant m} (\xi_s - \xi_r) \, \mathrm{d}\xi_0 \, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_m \end{aligned} \tag{3.13}$$

which says that $z_1, \ldots, z_m$ are the only distinct real non-negative zeros of $R(x)$ of odd multiplicity and therefore $p_j(x)R(x)$ has a constant sign, in contradiction to (3.12).

4. Conclusion

We have shown with the arguments of Ercolani and McLaughlin that if the weight $w(x, y)$ is such that $\det[w(x_i, y_j)]_{i,j=1,...,n} > 0$ for $x_1 < x_2 < \cdots < x_n$, $y_1 < y_2 < \cdots < y_n$ and moments $\int w(x, y)x^i y^j \, dx \, dy$ exist for all $i, j \geqslant 0$, then bi-orthogonal polynomials exist and their zeros are real, simple and lie in the respective supports of the weight $w(x, y)$. The same is true for a weight which is a convolution of several such weights.

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[3] For example, see Mehta M L 1991 *Random Matrices* (New York: Academic) appendix A.5

[4] For example, see Mehta M L 1989 *Matrix Theory* (Z.I. de Courtaboeuf, 91944 Les Ulis Cedex, France: Les Editions de Physique) sections 3.7 and 7.1.3